Integration durch Substitution der Integrationsvariablen

$$\int_a^b f(x)dx = \int_{\bar{g}(a)}^{\bar{g}(b)} f(g(t)) \cdot g'(t)\,dt$$

Logo!
x=g(t) und dx=g'(t)dt

Beispiel:

$$\int_0^4 \frac{2x}{\sqrt{9+x^2}}dx =$$

Substitution: $t = 9 + x^2 \quad \Rightarrow x = \sqrt{t-9}$

$$t'(x) = \frac{dt}{dx} = 2x \qquad \Rightarrow dx = \frac{dt}{2x}$$

$$\int_0^5 \frac{2x}{\sqrt{9+x^2}}dx = \int \frac{1}{\sqrt{t}} \cdot dt =$$

Integralgrenzen ?

Integration durch Substitution der Integrationsvariablen

Beispiel: $\int_0^4 \frac{2x}{\sqrt{9+x^2}}\,dx =$

Substitution: $t = 9 + x^2$

Logo!
Integralgrenzen:
$x = 0 \Rightarrow t = 9 \quad x = 4 \Rightarrow t = 25$

$$\Rightarrow x = \sqrt{t-9}$$

$$t'(x) = \frac{dt}{dx} = 2x \quad \Rightarrow dx = \frac{dt}{2x}$$

$$\int_0^4 \frac{2x}{\sqrt{9+x^2}}\,dx = \int_4^{25} \frac{1}{\sqrt{t}}\cdot dt = \int_4^{25} t^{-\frac{1}{2}}dt = \left[2\cdot t^{\frac{1}{2}}\right]_4^{25}$$

$$= \left[2\cdot\left(9+x^2\right)^{\frac{1}{2}}\right]_0^4 = \left[2\cdot\sqrt{9+x^2}\right]_0^4 = 10 - 6 = 4$$

Integration durch Substitution der Integrationsvariablen

$$\int_2^3 \frac{3}{\sqrt{x-1}}\,dx =$$

Substitution: $t = x - 1$

$$\Rightarrow x = t + 1$$

$$t'(x) = \frac{dt}{dx} = 1 \quad \Rightarrow dx = dt$$

$$\int_2^3 \frac{3}{\sqrt{x-1}}\,dx = \int \frac{3}{\sqrt{t}} \cdot dt = 3\int t^{-\frac{1}{2}}dt = 3\left[2 \cdot t^{\frac{1}{2}}\right]$$

$$= 3\left[2 \cdot (x-1)^{\frac{1}{2}}\right]_2^3 = \left[6 \cdot \sqrt{x-1}\right]_2^3$$

$$\int \frac{3}{\sqrt{x-1}}\,dx = 6 \cdot \sqrt{x-1} + c$$

Logo! Wenn ich die Substitution rückgängig mache, muss ich die Grenzen des t-Integrals nicht bestimmen!

Integration durch Substitution der Integrationsvariablen

$$\int_0^1 \sqrt{9-4x}\,dx =$$

t= 9-4x

Substitution: $t = 9 - 4x$ $\quad \Rightarrow x = \frac{9-t}{4}$

$$t'(x) = \frac{dt}{dx} = -4 \quad \Rightarrow dx = -\frac{1}{4}dt$$

$$\int_0^1 \sqrt{9-4x}\,dx = \int -\frac{1}{4}\sqrt{t}\,dt = -\frac{1}{4}\int t^{\frac{1}{2}}dt = -\frac{1}{4}\left[\frac{2}{3}\cdot t^{\frac{3}{2}}\right]$$

$$= -\frac{1}{4}\left[\frac{2}{3}\cdot\sqrt{(9-4x)^3}\right]_0^1 = -\frac{5\sqrt{5}}{6} + \frac{9}{2}$$

Integration durch Substitution der Integrationsvariablen

Beispiel: $\int_0^1 x(4x^2-1)^3\,dx =$

t= 4x²-1

Substitution: $t = 4x^2 - 1$

$$t'(x) = \frac{dt}{dx} = 8x \quad \Rightarrow dx = \frac{1}{8x}dt$$

$$\int_0^1 x(4x^2-1)^3\,dx = \int \frac{1}{8}t^3\,dt = \frac{1}{8}\int t^3 dt = \frac{1}{8}\left[\frac{1}{4}\cdot t^4\right]$$

$$= \frac{1}{32}\left[(4x^2-1)^4\right]_0^1 = \frac{81}{32} - \frac{1}{32} = \frac{5}{2}$$

Substitutionsverfahren:

1. Wähle eine geeignete Substitution:

   $t = \dots\dots$ oder $x = \dots\dots$

2. Verwandle damit das x-Integral in ein t-Integral:

   $$\frac{dt}{dx} = \dots\dots \Rightarrow dx = \frac{1}{\dots\dots} dt \qquad \frac{dx}{dt} = \dots\dots \Rightarrow dx = \dots\dots \cdot dt$$

3. Bestimme das t-Integral ,falls möglich

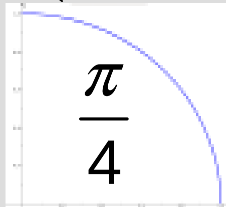
$t = \dots\dots$ oder
$x = \dots\dots$

$dx = \dots\dots$

$\int \dots\dots dt =$

Trickreiche Substitution

Beispiel: $\int_0^1 \sqrt{1-x^2}\,dx =$



Substitution: $x = \cos(t)$ $\Rightarrow t = \arccos(x)$

$$x'(x) = \frac{dx}{dt} = -\sin(t) \quad \Rightarrow dx = -\sin(t)dt$$

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_{\frac{\pi}{2}}^{0} \sqrt{1-\cos^2(t)} \cdot -\sin(t)dt = \quad -\int_{\frac{\pi}{2}}^{0} \sin^2(t)dt$$

$$\int \sin^2(t)dt = \int \underbrace{\sin(t)}_{u} \cdot \underbrace{\sin(t)}_{v'} dt = -\sin(t)\cdot\cos(t) + \int \cos^2(t)\,dt$$

$$= -\sin(t)\cdot\cos(t) + \int 1-\sin^2(t)\,dt \quad = -\sin(t)\cdot\cos(t) + \int 1-\sin^2(t)\,dt$$

Trickreiche Substitution

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_{\frac{\pi}{2}}^{0} \sqrt{1-\cos^2(t)} \cdot -\sin(t)dt = -\int_{\frac{\pi}{2}}^{0} \sin^2(t)dt = \frac{\pi}{4}$$

$$\int \sin^2(t)dt = -\sin(t)\cdot\cos(t) + \int (1-\sin^2(t))\,dt$$

$$-\sin(t)\cdot\cos(t) + t - \int \sin^2(t)\,dt$$

$$2\cdot\int \sin^2(t)dt = -\sin(t)\cdot\cos(t) + t$$

$$\int \sin^2(t)dt = \frac{t}{2} - \frac{1}{2}\sin(t)\cdot\cos(t)$$

$$\int \sin^2(t)dt = \int_{\frac{\pi}{2}}^{0} \sin^2(t)dt = \left[\frac{t}{2} - \frac{1}{2}\sin(t)\cdot\cos(t)\right]_{\frac{\pi}{2}}^{0} = -\frac{\pi}{4}$$

Trickreiche Substitution

Beispiel 2: $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx =$

I remember :

$\cos^2 x = 1 - \sin^2 x$

Substitution: $x = \sin(t)$ $\Rightarrow t = \arcsin(x)$

$$x'(t) = \frac{dx}{dt} = \cos(t) \qquad \Rightarrow dx = \cos(t)dt$$

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{\cos t}{\sqrt{1-\cos^2 t}}\,dt = \int_0^{\frac{\pi}{6}} 1dt = [t]_0^{\frac{\pi}{6}} = \frac{\pi}{6}$$

$$\sqrt{1-\cos^2 t} = |\cos t| = \cos t \quad \text{weil } \cos t \geq 0 \text{ für alle } x \in \left[0; \frac{\pi}{6}\right]$$